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A Simple Proof of A. F. Timan's Theorem

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1. INTRODUCTION

In 1951 Timan [8] proved the following well-known theorem:

THEOREM 1.1. *For every function f continuous on $[-1, 1]$ and any natural n there is an algebraic polynomial $p_n(f; \cdot)$ of degree n such that for all $t \in [-1, 1]$,*

$$|f(t) - p_n(f; t)| \leqslant M\omega\left(f; \frac{\sqrt{1-t^2}}{n} + \frac{|t|}{n^2}\right), \tag{1.1}$$

where M is an absolute constant and $\omega(f; \delta)$ is the modulus of continuity of f.

Today at least 20 different proofs of this theorem are known. A characteristic point of all these proofs is the construction of a sequence of algebraic polynomials for which inequality (1.1) is satisfied. To achieve this end, two different methods can be employed.

The first method starts from a sequence $(t_n(g; \cdot))_{n \in \mathbb{N}}$ of trigonometric polynomials of degree n, defined for the function $g(x) = f(\cos x)$ by a suitable convolution operator with positive even kernel. Then by means of the substitution $t = \cos x$ one obtains a sequence $(p_n(f; \cdot))_{n \in \mathbb{N}}$ of algebraic polynomials of degree n for which inequality (1.1) must be verified (cf., e.g., Timan [8] and Lorentz [4, pp. 65–69]).

The second method is to construct a sequence $(p_n(f; \cdot))_{n \in \mathbb{N}}$ of algebraic polynomials of degree n by means of a suitable interpolation process (cf. [2, 3, 6, 7, 9, 10]). If one compares these two methods, one realizes that the proofs using the second method are very complicated.

After surveying all the papers proving Timan's theorem, the question arises whether one can find a new proof with the two properties of being very short and simple, and of yielding information on the absolute constant M in (1.1). Some progress in this direction has been made by Saxena [7].

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THEOREM 1.2. *For every function f continuous on $[-1, 1]$ and any natural n there is an algebraic polynomial $A_n(f; \cdot)$ of degree $4n+2$ such that for all $t \in [-1, 1]$,*

$$|f(t) - A_n(f; t)| \leqslant 384 \left[\omega\left(f; \frac{\sqrt{1-t^2}}{n}\right) + \omega\left(f; \frac{|t|}{n^2}\right)\right]. \tag{1.2}$$

However, in our opinion the proof of this theorem is very complicated and much too long. Moreover, the constant in (1.2) is surely not the best one. Thus, the purpose of this paper is to give a good positive answer to the above question.

2. PRELIMINARIES

DEFINITION 2.1. For every function f continuous on $[-1, 1]$ and any natural n, the operator $G_{m(n)}$ is defined by

$$G_{m(n)}(f; t) := \frac{1}{\pi} \int_{-\pi}^{\pi} f(\cos(\arccos t + v))\, K_{m(n)}(v)\, dv, \tag{2.1}$$

where the kernel $K_{m(n)}$ is a trigonometric polynomial of degree $m(n)$ with the following properties:

(i) $K_{m(n)}$ is positive and even;

(ii) $\int_{-\pi}^{\pi} K_{m(n)}(v)\, dv = \pi$, i.e., $G_{m(n)}(1; t) = 1$ for $t \in [-1, 1]$;

(iii) $K_{m(n)}$ is an approximate identity, i.e.,

$$\lim_{n \to \infty} \{\max_{\delta \leqslant x \leqslant \pi} K_{m(n)}(x)\} = 0 \qquad \text{for} \quad 0 < \delta < \pi.$$

Remarks. (1) Let $f \in C[-1, 1]$. Then $G_{m(n)}(f; \cdot)$ is an algebraic polynomial of degree $m(n)$.

(2) We can write

$$K_{m(n)}(v) = \tfrac{1}{2} + \sum_{k=1}^{m(n)} \rho_{k,m(n)} \cos kv, \qquad v \in [-\pi, \pi]. \tag{2.2}$$

LEMMA 2.2. *We have*

$$G_{m(n)}((v-t)^2; t) = t^2[\tfrac{3}{2} - 2\rho_{1,m(n)} + \tfrac{1}{2}\rho_{2,m(n)}] + (1-t^2)\,\tfrac{1}{2}(1-\rho_{2,m(n)}) \qquad \textit{for all} \quad t \in [-1, 1]. \tag{2.3}$$

Proof. For $t \in [-1, 1]$ we have

$$\begin{aligned}
G_{m(n)}((v-t)^2; t) &= \frac{1}{\pi} \int_{-\pi}^{\pi} (\cos(\arccos t + v) - t)^2 K_{m(n)}(v)\, dv \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} (t \cos v - \sqrt{1-t^2} \sin v - t)^2 K_{m(n)}(v)\, dv \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} [t^2(1-\cos v)^2 + (1-t^2) \sin^2 v \\
&\quad + t\sqrt{1-t^2}\, 2(1-\cos v) \sin v]\, K_{m(n)}(v)\, dv \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} [t^2(1 - 2\cos v + \cos^2 v) \\
&\quad + (1-t^2) \sin^2 v]\, K_{m(n)}(v)\, dv \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} [t^2(1 - 2\cos v + \tfrac{1}{2}(1 + \cos 2v)) \\
&\quad + (1-t^2)\tfrac{1}{2}(1 - \cos 2v)]\, K_{m(n)}(v)\, dv \\
&= t^2[\tfrac{3}{2} - 2\rho_{1,m(n)} + \tfrac{1}{2}\rho_{2,m(n)}] \\
&\quad + (1-t^2)\tfrac{1}{2}(1 - \rho_{2m(n)}). \qquad \blacksquare
\end{aligned}$$

THEOREM 2.3. *Let $(L_n)_{n \in \mathbb{N}}$ be a sequence of positive linear operators mapping $C[-1, 1]$ into $C[-1, 1]$ with the property*

$$L_n(1; t) = 1, \qquad t \in [-1, 1].$$

Then for all $f \in C[-1, 1]$ and all $t \in [-1, 1]$ we have

$$|L_n(f; t) - f(t)| \leqslant 2\omega(f; \alpha_n(t)), \tag{2.4}$$

where $\alpha_n(t)$ is defined by

$$\alpha_n^2(t) = L_n((v-t)^2; t), \qquad t \in [-1, 1].$$

A proof of Theorem 2.3 is given in [1, p. 28].

3. RESULTS

THEOREM 3.1. *For every function $f \in C[-1, 1]$ and any natural n there is an algebraic polynomial $H_n(f; \cdot)$ of degree $3n - 3$ such that for all $t \in [-1, 1]$*

$$|f(t)-H_n(f;t)|\leqslant 2\omega\left(f;\sqrt{\frac{30}{11}}\frac{|t|}{n^2}+\sqrt{\frac{20}{11}}\frac{\sqrt{1-t^2}}{n}\right)$$
$$\leqslant 4\left[\omega\left(f;\frac{|t|}{n^2}\right)+\omega\left(f;\frac{\sqrt{1-t^2}}{n}\right)\right]. \tag{3.1}$$

Proof. Let us consider the operator

$$H_n(f;t)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\cos(\arccos t+v))\,K_{3n-3}(v)\,dv$$

with the kernel

$$K_{3n-3}(v)=\frac{10}{n(11n^4+5n^2+4)}\left(\frac{\sin(nv/2)}{\sin(v/2)}\right)^6$$
$$=\tfrac{1}{2}+\sum_{k=1}^{3n-3}\rho_{k,3n-3}\cos kv.$$

Then, by [5, p. 14], we have

$$\rho_{1,3n-3}=\frac{11n^4-5n^2-6}{11n^4+5n^2+4},\qquad \rho_{2,3n-3}=\frac{11n^4-35n^2+24}{11n^4+5n^2+4},$$

and thus by means of Lemma 2.2

$$H_n((v-t)^2;t)=\frac{30t^2}{11n^4+5n^2+4}+\frac{(20n^2-10)(1-t^2)}{11n^4+5n^2+4}$$
$$\leqslant\frac{30t^2}{11n^4}+\frac{20(1-t^2)}{11n^2}$$
$$\leqslant\left\{\sqrt{\frac{30}{11}}\frac{|t|}{n^2}+\sqrt{\frac{20}{11}}\frac{\sqrt{1-t^2}}{n}\right\}^2. \tag{3.2}$$

From (3.2) the result of Theorem 3.1 follows immediately by an application of Theorem 2.3. ∎

THEOREM 3.2. *For every function $f\in C[-1,1]$ and any natural n there is an algebraic polynomial $\bar{H}_n(f;\cdot)$ of degree $2n-2$ such that for all $t\in[-1,1]$*

$$|f(t)-\bar{H}_n(f;t)|\leqslant 2\omega\left(f;\frac{3}{2}\frac{|t|}{n^{3/2}}+\sqrt{3}\,\frac{\sqrt{1-t^2}}{n}\right)$$
$$\leqslant 4\omega\left(f;\frac{|t|}{n^{3/2}}\right)+4\omega\left(f;\frac{\sqrt{t-t^2}}{n}\right). \tag{3.3}$$

Proof. Let K_{2n-2} be Jackson's kernel

$$K_{2n-2}(v) = \frac{3}{2n(2n^2+1)} \left(\frac{\sin(nv/2)}{\sin(v/2)} \right)^4 = \tfrac{1}{2} + \sum_{k=1}^{2n-2} \rho_{k,2n-2} \cos kv$$

with

$$\rho_{1,2n-2} = \frac{2n^3 - 2n}{2n^3 + n}, \qquad \rho_{2,2n-2} = \frac{2n^3 - 11n + 9}{2n^2 + n}.$$

Thus, for the corresponding operator of type (2.1) we have

$$\begin{aligned} \bar{H}_n((v-t)^2; t) &= \frac{9t^2}{4n^2 + 2n} + \frac{(12n-9)(1-t^2)}{4n^3 + 2n} \\ &\leqslant \left\{ \frac{3}{2} \frac{|t|}{n^{3/2}} + \sqrt{3} \frac{\sqrt{1-t^2}}{n} \right\}^2. \quad \blacksquare \qquad (3.4) \end{aligned}$$

REFERENCES

1. R. A. DEVORE, "The Approximation of Continuous Functions by Positive Linear Operators," Lecture Notes in Mathematics No. 293, Springer-Verlag, Berlin, 1972.
2. G. FREUD AND P. VERTESI, A new proof of A. F. Timan's approximation theorem, *Studia Sci. Math. Hungar.* **2** (1967), 403–414.
3. G. FREUD AND A. SHARMA, Some good sequences of interpolatory polynomials, *Canad. J. Math.* **26** (1974), 233–246.
4. G. G. LORENTZ, "Approximation of Functions," Holt, Rinehart & Winston, New York, 1966.
5. Y. MATSUOKA, On the degree of approximation of functions by some positive linear operators, *Sci. Rep. Kagoshima Univ.* **9** (1960), 11–16.
6. T. M. MILLS AND A. K. VARMA, A new proof of A. F. Timan's approximation theorem, *Israel J. Math.* **18** (1974), 39–44.
7. R. B. SAXENA, The approximation of continuous functions by interpolatory polynomials, *Proc. Math. Inst. Bulgar. Akad. Sci.* **12** (1970), 97–105.
8. A. F. TIMAN, Strengthening of Jackson's theorem on the best approximation of continuous functions on a finite segment of the real axis, *Dokl. Akad. Nauk SSSR* **78** (1951), 17–20.
9. A. K. VARMA, A new proof of A. F. Timan's approximation theorem, II, *J. Approx. Theory* **18** (1976), 57–62.
10. P. VERTESI AND O. KIS, On a new interpolation process, *Ann. Univ. Sci. Budapest* **10** (1967), 117–128.